Semi-symmetry properties of the tangent bundle with a pseudo-Riemannian metric

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Abstract. In this note, we consider the tangent bundle TM equipped with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g). We investigate semi-symmetry properties of the tangent bundle TM with respect to the Levi-Civita connection $\widetilde{\nabla}$ and a metric connecton $\widehat{\nabla}$ with torsion.

Keywords: metric connection, pseudo-Riemannian metric, Ricci semi-symmetry, semi-symmetry, tangent bundle.

1. Introduction

A Riemannian manifold (M, g) is said to be locally symmetric if the Riemannian curvature tensor R is parallel with respect to the Levi-Civita connection ∇, i.e., $\nabla R = 0$. A natural generalization of the notion of local symmetry is semi-symmetry. A semi-symmetric space is a (pseudo-) Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0$$

for all vector fields X and Y on M, where $R(X, Y)$ is a linear operator acting as a derivation on the curvature tensor R. This class of spaces was first studied by E. Cartan. Nevertheless, N. S. Sinjukov first used the name "semi-symmetric spaces" for manifolds satisfying the above curvature condition [5]. Later, Z. I. Szabo gave the full local and global classification of semi-symmetric spaces

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[6, 7]. A (pseudo-)Riemannian manifold (M, g) is called Ricci semi-symmetric if the following condition is satisfied:

$$R(X, Y) \cdot Ric = 0,$$

where Ric is the Ricci tensor of (M, g). It is clear that any semi-symmetric manifold is Ricci semi-symmetric.

In this paper, we consider the tangent bundle TM with a pseudo-Riemannian metric $\widetilde{g}$. Semi-symmetry and Ricci semi-symmetry are investigated first for the Levi-Civita connection then for a metric connection with a non-zero torsion tensor which is in a special form.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : TMarrowM$ its tangent bundle with fibres the tangent spaces to M. Then TM is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely a system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = y^i)$, $\bar{i} = n + i = n + 1, ..., 2n$, where (y^i) is the cartesian coordinates in each tangent space T_pM at $p \in M$ with respect to the natural base $\{\frac{\partial}{\partial x^i}|_p.\}$, p being an arbitrary point in U whose coordinates are (x^i).

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^VX$ and the horizontal lift ${}^HX$ of X are given, with respect to the induced coordinates, by

$$ {}^VX = X^i \partial_{\bar{i}}, \tag{2.1}$$

$$ {}^HX = X^i \partial_i - y^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [9].

In each coordinate neighborhood U of M, we put $X = X_j = \frac{\partial}{\partial x^j}$ in (2.1) and (2.2), we get in each induced coordinate neighborhood $\pi^{-1}(U)$ of TM a frame field which consists of the following $2n$ linearly independent vector fields: [9]

$$\begin{cases} E_j = {}^H(X_j) = \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}}, \\ E_{\overline{j}} = {}^V(X_j) = \partial_{\overline{j}}. \end{cases}$$

We will mark the adapted frame as $\{E_\beta\} = \{E_j, E_{\overline{j}}\}$. The Lie brackets of the adapted frame are as follows: [9]

$$\begin{cases} [E_i, E_j] = y^s R_{jis}{}^h E_{\overline{h}}, \\ \left[E_i, E_{\overline{j}}\right] = \Gamma^h_{ij} E_{\overline{h}}, \\ \left[E_{\overline{i}}, E_{\overline{j}}\right] = 0 \end{cases} \tag{2.3}$$

where ${R_{hji}}^s$ denote components of the curvature tensor field R of g.

3. Semi-symmetry properties of the tangent bundle with respect to the Levi-Civita connection $\widetilde{\nabla}$

Let (M, g) be a Riemannian manifold. A pseudo-Riemannian metric $\widetilde{g}$ on the tangent bundle TM over (M, g) is defined by

$$\begin{cases} \widetilde{g}(^HX, ^HY) = a(X, Y) + a(Y, X), \\ \widetilde{g}(^VX, ^HY) = \widetilde{g}(^HX, ^VY) = g(X, Y), \\ \widetilde{g}(^VX, ^VY) = 0 \end{cases}$$

for all vector fields X and Y on M, where a is a $(0, 2)-$tensor field on M [4]. The pseudo-Riemannian metric $\widetilde{g}$ is expressed in the adapted local frame by

$$\widetilde{g} = (\widetilde{g}_{\alpha\beta}) = \begin{pmatrix} a_{ij} + a_{ji} & g_{ij} \\ g_{ij} & 0 \end{pmatrix},$$

where g_{ij} and a_{ij} respectively are the local components of g and a. If the $(0, 2)-$tensor field a is symmetric, then the metric $\widetilde{g}$ becomes the metric known as synectic lift metric in the literature. The synectic lift metric was firstly considered in [8] and then studied by some authors [1, 2].

For the Levi-Civita connection $\widetilde{\nabla}$ of the pseudo-Riemannian metric $\widetilde{g}$, we have:

Proposition 1. *The Levi-Civita connection $\widetilde{\nabla}$ of $(TM, \widetilde{g})$ is given by*

$$\begin{cases} \widetilde{\nabla}_{E_i} E_j = \Gamma^h_{ij} E_h + \{y^s {R_{sij}}^h + g^{mh}(\nabla_i c_{mj} + \nabla_j c_{im} - \nabla_m c_{ij})\} E_{\overline{h}}, \\ \widetilde{\nabla}_{E_i} E_{\overline{j}} = \Gamma^h_{ij} E_{\overline{h}}, \\ \widetilde{\nabla}_{E_{\overline{i}}} E_j = 0, \ \widetilde{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0, \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ^h_{ij} and ${R_{hji}}^s$ respectively denote components of the Levi-Civita connection ∇ and the curvature tensor field R of g on M and c_{ij} denotes the symmetric part of a_{ij}, i.e., $c_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ (see [4]).

Proof. The Koszul formula for a connection $\widetilde{\nabla}$ is given by

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) &= \widetilde{X}(\widetilde{g}(\widetilde{Y}, \widetilde{Z})) + \widetilde{Y}(\widetilde{g}(\widetilde{Z}, \widetilde{X})) - \widetilde{Z}(\widetilde{g}(\widetilde{X}, \widetilde{Y})) \\ -\widetilde{g}(\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]) &+ \widetilde{g}(\widetilde{Y}, [\widetilde{Z}, \widetilde{X}]) + \widetilde{g}(\widetilde{Z}, [\widetilde{X}, \widetilde{Y}]) \end{aligned}$$

for all vector fields $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ on TM. Using the Koszul formula for pairs $\widetilde{X} = E_i, E_{\overline{i}}$ and $\widetilde{Y} = E_j, E_{\overline{j}}$ and $\widetilde{Z} = E_k, E_{\overline{k}}$, the formulas (2.3) and the first Bianchi identity for R, we obtain the Christoffel symbols of the Levi-Civita connection $\widetilde{\nabla}$ of $(TM, \widetilde{g})$. We omit standart calculations. □

For the Riemannian curvature tensor $\widetilde{R}$ of the pseudo-Riemannian metric $\widetilde{g}$, we have:

Proposition 2. *The Riemannian curvature tensor $\widetilde{R}$ of $(TM, \widetilde{g})$ is given as follows:*

$$\begin{aligned}
\widetilde{R}(E_i, E_j)E_k &= R_{ijk}{}^h E_h + \{y^s(\nabla_i R_{sjk}{}^h - \nabla_j R_{sik}{}^h) \\
&\quad + g^{mh}(\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) - \nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) \\
&\quad - R_{ijm}{}^l c_{lk} - R_{ijk}{}^l c_{ml})\} E_{\overline{h}}, \\
\widetilde{R}(E_i, E_j)E_{\overline{k}} &= R_{ijk}{}^h E_{\overline{h}}, \\
\widetilde{R}(E_i, E_{\overline{j}})E_k &= R_{ijk}{}^h E_{\overline{h}}, \\
\widetilde{R}(E_{\overline{i}}, E_j)E_k &= R_{ijk}{}^h E_{\overline{h}}, \\
\widetilde{R}(E_{\overline{i}}, E_{\overline{j}})E_k &= 0, \ \widetilde{R}(E_{\overline{i}}, E_j)E_{\overline{k}} = 0, \\
\widetilde{R}(E_i, E_{\overline{j}})E_{\overline{k}} &= 0, \ \widetilde{R}(E_{\overline{i}}, E_{\overline{j}})E_{\overline{k}} = 0
\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$ (see also [4]).

A semi-Riemannian manifold (M, g), $n = \dim(M) \geq 3$, is said to be semi-symmetric [6] if its curvature tensor R satisfies the condition

$$\begin{aligned}
&((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^n = \nabla_i \nabla_j R_{klm}{}^n - \nabla_j \nabla_i R_{klm}{}^n \\
&= R_{ijp}{}^n R_{klm}{}^p - R_{ijk}{}^p R_{plm}{}^n - R_{ijl}{}^p R_{kpm}{}^n - R_{ijm}{}^p R_{klp}{}^n = 0
\end{aligned}$$

and Ricci semi-symmetric if its Ricci tensor satisfies the condition

$$((R(X,Y)\cdot Ric)(Z,W))_{ijkl} = \nabla_i \nabla_j R_{kl} - \nabla_j \nabla_i R_{kl} = R_{ijk}{}^p R_{pl} + R_{ijl}{}^p R_{kp} = 0.$$

Note that a locally symmetric manifold is obviously semi-symmetric, but in general the converse is not true.

Theorem 1. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g). We assume that $\nabla_i R_{sjk}{}^h - \nabla_j R_{sik}{}^h = 0$ and $\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) - \nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^l c_{lk} - R_{ijk}{}^l c_{ml} = 0$, where R is the curvature tensor of the Levi-Civita connection ∇ of g. Then the tangent bundle $(TM, \widetilde{g})$ is semi-symmetric if and only if the Riemannian manifold (M, g) is semi-symmetric.*

Proof. We consider the conditions $(\widetilde{R}(\widetilde{X}, \widetilde{Y}) \cdot \widetilde{R})(\widetilde{Z}, \widetilde{W})\widetilde{U} = 0$ for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}$ and $\widetilde{U}$ on TM. In the adapted frame $\{E_\beta\}$, the tensor $(\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{R})(\widetilde{Z}, \widetilde{W})\widetilde{U}$ is locally expressed as follows:

$$\begin{aligned}
(3.1) \qquad & ((\widetilde{R}(\widetilde{X}, \widetilde{Y}) \cdot \widetilde{R})(\widetilde{Z}, \widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^\varepsilon \\
&= \widetilde{R}_{\alpha\beta\tau}{}^\varepsilon \widetilde{R}_{\gamma\theta\sigma}{}^\tau - \widetilde{R}_{\alpha\beta\gamma}{}^\tau \widetilde{R}_{\tau\theta\sigma}{}^\varepsilon - \widetilde{R}_{\alpha\beta\theta}{}^\tau \widetilde{R}_{\gamma\tau\sigma}{}^\varepsilon - \widetilde{R}_{\alpha\beta\sigma}{}^\tau \widetilde{R}_{\gamma\theta\tau}{}^\varepsilon.
\end{aligned}$$

Putting $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = \overline{m}, \varepsilon = \overline{h}$ in (3.1), it follows that

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijkl\overline{m}}{}^{\overline{h}} \\
=\;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{kl\overline{m}}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{kl\overline{m}}{}^{\overline{p}} - \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{pl\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}l\overline{m}}{}^{\overline{h}} \\
&-\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kp\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{p}\widetilde{R}_{klp}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{h}} \\
=\;& R_{ijp}{}^{h}R_{klm}{}^{p} - R_{ijk}{}^{p}R_{plm}{}^{h} - R_{ijl}{}^{p}R_{kpm}{}^{h} - R_{ijm}{}^{p}R_{klp}{}^{h} \\
=\;& ((R(X,Y)\cdot R)(Z,W)U)_{ijklm}{}^{h}.
\end{aligned} \tag{3.2}$$

Putting $\alpha = i, \beta = j, \gamma = \overline{k}, \theta = \overline{l}, \sigma = \overline{m}, \varepsilon = \overline{n}$ in (3.1), we get

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ij\overline{k}\overline{l}\overline{m}}{}^{\overline{h}} \\
=\;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{\overline{k}\overline{l}\overline{m}}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{\overline{k}\overline{l}\overline{m}}{}^{\overline{p}} - \widetilde{R}_{ij\overline{k}}{}^{p}\widetilde{R}_{p\overline{l}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{k}}{}^{\overline{p}}\widetilde{R}_{\overline{p}\overline{l}\overline{m}}{}^{\overline{h}} \\
&-\widetilde{R}_{ij\overline{l}}{}^{p}\widetilde{R}_{\overline{k}p\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{l}}{}^{\overline{p}}\widetilde{R}_{\overline{k}\overline{p}\overline{m}}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{p}\widetilde{R}_{\overline{k}\overline{l}p}{}^{\overline{h}} - \widetilde{R}_{ij\overline{m}}{}^{\overline{p}}\widetilde{R}_{\overline{k}\overline{l}\overline{p}}{}^{\overline{h}} \\
=\;& 0.
\end{aligned} \tag{3.3}$$

Putting $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = \overline{h}$ in (3.1), we obtain

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijklm}{}^{\overline{h}} \\
=\;& \widetilde{R}_{ijp}{}^{\overline{h}}\widetilde{R}_{klm}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{h}}\widetilde{R}_{klm}{}^{\overline{p}} - \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{plm}{}^{\overline{h}} - \widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}lm}{}^{\overline{h}} \\
&-\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kpm}{}^{\overline{h}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}m}{}^{\overline{h}} - \widetilde{R}_{ijm}{}^{p}\widetilde{R}_{klp}{}^{\overline{h}} - \widetilde{R}_{ijm}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{h}}.
\end{aligned} \tag{3.4}$$

Let us assume that

$$\begin{aligned}
\widetilde{R}_{ijk}{}^{\overline{h}} &= y^s(\nabla_i R_{sjk}{}^{h} - \nabla_j R_{sik}{}^{h}) + g^{mh}(\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) \\
&\quad -\nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^{l}c_{lk} - R_{ijk}{}^{l}c_{ml}) \\
&= 0.
\end{aligned} \tag{3.5}$$

On differentiating the above equation with respect to $\partial_{\overline{l}}$, we obtain

$$\nabla_i R_{ljk}{}^{h} - \nabla_j R_{lik}{}^{h} = 0. \tag{3.6}$$

In the case, (3.5) reduces the following form

$$\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) - \nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^{l}c_{lk} - R_{ijk}{}^{l}c_{ml} = 0. \tag{3.7}$$

The other coefficients of $(\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}$ reduce to one of (3.2) and (3.3). Hence the proof follows from (3.2)-(3.7). □

Let $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ denote the components of Ricci tensor of $(TM, \widetilde{g})$. From the Proposition 2, we have only one non-zero component, i.e., $\widetilde{R}_{ij} = 2R_{ij}$ and other components are zero. The tensor $(\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{Ric})(\widetilde{Z},\widetilde{W})$ has components

$$((\widetilde{R}(\widetilde{X},\widetilde{Y})\cdot\widetilde{Ric})(\widetilde{Z},\widetilde{W}))_{\alpha\beta\gamma\theta} = \widetilde{R}_{\alpha\beta\gamma}{}^{\varepsilon}\widetilde{R}_{\varepsilon\theta} + \widetilde{R}_{\alpha\beta\theta}{}^{\varepsilon}\widetilde{R}_{\gamma\varepsilon}$$

with respect to the adapted frame $\{E_\beta\}$. The following corollary is a direct consequence of Theorem 1.

Corollary 1. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g). Then the tangent bundle $(TM, \widetilde{g})$ is Ricci semi-symmetric if and only if the Riemannian manifold (M, g) is Ricci semi-symmetric.*

4. Semi-symmetry properties of the tangent bundle with respect to the metric connection ∇ with torsion

H. A. Hayden introduced the idea of metric connection with torsion on a Riemannian manifold [3]. A linear connection ∇ on a Riemannian manifold (M, g) is called a metric connection with repect to g if $\nabla g = 0$ and its torsion tensor is non-zero. In this section we consider a metric connection on $(TM, \widetilde{g})$ with a non-zero torsion tensor.

Let $\widehat{\nabla}$ be a metric connection on TM and its torsion tensor $\widehat{T}$ have only one non-zero component, i.e., $\widehat{T}\,{}_{ij}^{\overline{h}} = y^s R_{ijs}{}^h$ and other components are zero. The torsion tensor $\widehat{T}$ is very specific, which is actually related to the Lie bracket of $[E_i, E_j]$ (see the formulas (2.3)). On following the method shown in ([9], p.151-152), we get:

Proposition 3. *The metric connection $\widehat{\nabla}$ of $(TM, \widetilde{g})$ is given by*

$$\begin{cases} \widehat{\nabla}_{E_i} E_j = \Gamma_{ij}^h E_h + g^{mh}(\nabla_i c_{mj} + \nabla_j c_{im} - \nabla_m c_{ij}) E_{\overline{h}}, \\ \widehat{\nabla}_{E_i} E_{\overline{j}} = \Gamma_{ij}^h E_{\overline{h}}, \\ \widehat{\nabla}_{E_{\overline{i}}} E_j = 0, \ \widehat{\nabla}_{E_{\overline{i}}} E_{\overline{j}} = 0, \end{cases}$$

with respect to the adapted frame $\{E_\beta\}$, where Γ_{ij}^h denote components of the Levi-Civita connection ∇ of g on M and c_{ij} denotes the symmetric part of a_{ij}, i.e., $c_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ (see also [1]).

For the curvature tensor $\widehat{R}$ of the metric connection $\widehat{\nabla}$, we have:

Proposition 4. *The curvature tensor $\widehat{R}$ of the metric connection $\widehat{\nabla}$ is given by*

$$\begin{aligned} \widehat{R}(E_i, E_j)E_k &= R_{ijk}{}^h E_h + g^{mh}(\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) \\ &\quad -\nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^l c_{lk} - R_{ijk}{}^l c_{ml})\} E_{\overline{h}}, \\ \widehat{R}(E_i, E_j)E_{\overline{k}} &= R_{ijk}{}^h E_{\overline{h}}, \ \widehat{R}(E_i, E_{\overline{j}})E_k = 0, \ \widehat{R}(E_{\overline{i}}, E_j)E_k = 0, \\ \widehat{R}(E_{\overline{i}}, E_{\overline{j}})E_k &= 0, \widehat{R}(E_{\overline{i}}, E_j)E_{\overline{k}} = 0, \ \widehat{R}(E_i, E_{\overline{j}})E_{\overline{k}} = 0, \\ \widehat{R}(E_{\overline{i}}, E_{\overline{j}})E_{\overline{k}} &= 0 \end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$, where $R_{ijk}{}^h$ denote components of the curvature tensor field R of g on M.

Theorem 2. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g) and $\widehat{\nabla}$ be the metric connection with* *torsion of* $(TM, \widetilde{g})$. *We assume that* $\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) - \nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^l c_{lk} - R_{ijk}{}^l c_{ml} = 0$, *where* R *is the curvature tensor of the Levi-Civita connection* ∇ *of* g. *Then the tangent bundle* $(TM, \widetilde{g})$ *is semi-symmetric with respect to the metric connection* $\widehat{\nabla}$ *if and only if the Riemannian manifold* (M, g) *is semi-symmetric.*

Proof. We calculate

$$
\begin{aligned}
(4.1)\qquad & ((\widehat{R}(\widetilde{X}, \widetilde{Y}) \cdot \widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon} \\
= {} & \widehat{R}_{\alpha\beta\tau}{}^{\varepsilon}\, \widehat{R}_{\gamma\theta\sigma}{}^{\tau} - \widehat{R}_{\alpha\beta\gamma}{}^{\tau}\, \widehat{R}_{\tau\theta\sigma}{}^{\varepsilon} - \widehat{R}_{\alpha\beta\theta}{}^{\tau}\, \widehat{R}_{\gamma\tau\sigma}{}^{\varepsilon} - \widehat{R}_{\alpha\beta\sigma}{}^{\tau}\, \widehat{R}_{\gamma\theta\tau}{}^{\varepsilon}
\end{aligned}
$$

for all cases $\alpha = (i, \bar{i})$, $\beta = (j, \bar{j})$, $\gamma = (k, \bar{k})$, $\theta = (l, \bar{l})$, $\sigma = (m, \overline{m})$ and $\varepsilon = (h, \bar{h})$.

The cases $\alpha = i$, $\beta = j$, $\gamma = k$, $\theta = l$, $\sigma = m$, $\varepsilon = h$ and $\alpha = i$, $\beta = j$, $\gamma = k$, $\theta = l$, $\sigma = \overline{m}$, $\varepsilon = \bar{h}$ in (4.1), we respectively get

$$
\begin{aligned}
& ((\widehat{R}(\widetilde{X}, \widetilde{Y}) \cdot \widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U})_{ijklm}{}^{h} \\
= {} & \widehat{R}_{ijp}{}^{h}\, \widehat{R}_{klm}{}^{p} + \widehat{R}_{ij\bar{p}}{}^{h}\, \widehat{R}_{klm}{}^{\bar{p}} - \widehat{R}_{ijk}{}^{p}\, \widehat{R}_{plm}{}^{h} - \widehat{R}_{ijk}{}^{\bar{p}}\, \widehat{R}_{\bar{p}lm}{}^{h} \\
& - \widehat{R}_{ijl}{}^{p}\, \widehat{R}_{kpm}{}^{h} - \widehat{R}_{ijl}{}^{\bar{p}}\, \widehat{R}_{k\bar{p}m}{}^{h} - \widehat{R}_{ijm}{}^{p}\, \widehat{R}_{klp}{}^{h} - \widehat{R}_{ijm}{}^{\bar{p}}\, \widehat{R}_{kl\bar{p}}{}^{h} \\
= {} & R_{ijp}{}^{h} R_{klm}{}^{p} - R_{ijk}{}^{p} R_{plm}{}^{h} - R_{ijl}{}^{p} R_{kpm}{}^{h} - R_{ijm}{}^{p} R_{klp}{}^{h} \\
= {} & ((R(X, Y) \cdot R)(Z, W)U)_{ijklm}{}^{h}
\end{aligned}
$$

and similarly

$$
((\widehat{R}(\widetilde{X}, \widetilde{Y}) \cdot \widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U})_{ijkl\overline{m}}{}^{\bar{h}} = ((R(X, Y) \cdot R)(Z, W)U)_{ijklm}{}^{h}.
$$

The case $\alpha = i$, $\beta = j$, $\gamma = k$, $\theta = l$, $\sigma = m$, $\varepsilon = \bar{h}$ in (4.1), we obtain

$$
\begin{aligned}
& ((\widehat{R}(\widetilde{X}, \widetilde{Y})\widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U})_{ijklm}{}^{\bar{h}} \\
= {} & \widehat{R}_{ijp}{}^{\bar{h}}\, \widehat{R}_{klm}{}^{p} + \widehat{R}_{ij\bar{p}}{}^{\bar{h}}\, \widehat{R}_{klm}{}^{\bar{p}} - \widehat{R}_{ijk}{}^{p}\, \widehat{R}_{plm}{}^{\bar{h}} - \widehat{R}_{ijk}{}^{\bar{p}}\, \widehat{R}_{\bar{p}lm}{}^{\bar{h}} \\
& - \widehat{R}_{ijl}{}^{p}\, \widehat{R}_{kpm}{}^{\bar{h}} - \widehat{R}_{ijl}{}^{\bar{p}}\, \widehat{R}_{k\bar{p}m}{}^{\bar{h}} - \widehat{R}_{ijm}{}^{p}\, \widehat{R}_{klp}{}^{\bar{h}} - \widehat{R}_{ijm}{}^{\bar{p}}\, \widehat{R}_{kl\bar{p}}{}^{\bar{h}} \\
= {} & \widehat{R}_{ijp}{}^{\bar{h}}\, \widehat{R}_{klm}{}^{p} + \widehat{R}_{ij\bar{p}}{}^{\bar{h}}\, \widehat{R}_{klm}{}^{\bar{p}} - \widehat{R}_{ijk}{}^{p}\, \widehat{R}_{plm}{}^{\bar{h}} - \widehat{R}_{ijl}{}^{p}\, \widehat{R}_{kpm}{}^{\bar{h}} \\
& - \widehat{R}_{ijm}{}^{p}\, \widehat{R}_{klp}{}^{\bar{h}} - \widehat{R}_{ijm}{}^{\bar{p}}\, \widehat{R}_{kl\bar{p}}{}^{\bar{h}}.
\end{aligned}
$$

For the other cases, $(\widehat{R}(\widetilde{X}, \widetilde{Y}) \cdot \widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U} = 0$. If $\widehat{R}_{ijp}{}^{\bar{h}} = 0$, i.e., $\nabla_i(\nabla_k c_{jm} - \nabla_m c_{jk}) - \nabla_j(\nabla_k c_{im} - \nabla_m c_{ik}) - R_{ijm}{}^l c_{lk} - R_{ijk}{}^l c_{ml} = 0$, then $(\widehat{R}(\widetilde{X}, \widetilde{Y}) \cdot \widehat{R})(\widetilde{Z}, \widetilde{W})\widetilde{U} = 0$ if and only if $(R(X, Y) \cdot R)(Z, W)U = 0$. This complets the proof. □

On computing the contracted curvature tensor (Ricci tensor) $\widehat{R}_{\gamma\beta}$, the only non-zero component is as follows: $\widehat{R}_{ij} = R_{ij}$ (see also [1]). As a consequence of the Theorem 3, we obtain:

Corollary 2. *Let TM be the tangent bundle with a pseudo-Riemannian metric $\widetilde{g}$ over a Riemannian manifold (M, g) and $\widehat{\nabla}$ be the metric connection of $(TM, \widetilde{g})$. Then the tangent bundle $(TM, \widetilde{g})$ is Ricci semi-symmetric with respect to the metric connection $\widehat{\nabla}$ if and only if the Riemannian manifold (M, g) is Ricci semi-symmetric.*

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